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THE EFFECT OF VARIATIONS IN THE CREEP EXPONENT ON THE BUCKLING OF CIRCULAR CYLINDRICAL SHELLS†

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Abstract—The senior author solved the problem of axially symmetrical creep buckling of thin circular cylindrical shells subjected to uniform axial compression. In that analysis the constitutive equation was a power law, and the exponent was taken to be equal to three. The purpose of this work was to extend the solution to a range of values of the creep exponent, n. To cope with the increasing algebraic complexity, a digital computer was employed in two ways: to generate the set of equations symbolically, and then to solve these equations. The computer programs were used to generate numerical solutions for the cases in which n was equal to 3, 5, 7 and 9. Two simple extrapolation techniques were then employed to obtain approximate solutions to the critical time problem for values of n up to 29.

NOTATION

a	radius of shell (see Fig. 1)
C_k	coefficients in series for axial stress in inner face
C_k^*	coefficients in series for axial stress in outer face
d	distance between faces of equivalent sandwich wall (see Fig. 2)
D_k	coefficients in series for hoop stress in inner face
D_k^*	coefficients in series for hoop stress in outer face
E	Young's modulus of elasticity
f_0, f_1	coefficients in expression for radial displacements [equation (1)]
$F1 = f_1/d$	nondimensional radial displacement coefficient (used in computer programming)
n	exponent in uniaxial creep formula
P	compressive load
$R = \pi^2 ad/\lambda^2$	dimensionless wavelength parameter
t	time measured from instant of load application
t_E	Euler time defined in equation (8)
t^*	wall thickness of real shell
w	radial displacement (see Fig. 1)
x	axial coordinate
$y = f_1/d$	nondimensional radial displacement
$\alpha = \pi x/\lambda$	
ε_E	Euler strain defined in equation (7)
$\dot{\varepsilon}_{\text{nom}}$	creep rate of material under uniaxial applied stress σ
λ	half wave length of buckle
$\sigma = P/2\pi a$	applied compressive stress
σ_{cr}	classical critical stress of axially compressed thin shell
σ_h	circumferential (hoop) stress
σ_x	axial stress
$\tau = t/t_E$	nondimensional time
Subscripts	
f	final
i	inner face
i	initial
0	outer face
Superscript	
	dot indicates differentiation with respect to time t

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INTRODUCTION

THE problem of creep buckling has received considerable attention during the past several years [1–4]. In particular, the creep buckling of circular cylindrical shells has been studied at Stanford [5–7]. In the senior author's paper, [5], the buckling was assumed to be axially symmetrical and a closed form solution for the critical time, i.e. the time at which the deformations become unbounded, was obtained. The constitutive equation was a power law, and the exponent was assumed to be equal to 3. The purpose of this paper is to find solutions for a range of values of the creep exponent, including nonintegral values.

One effect of increasing the value of the exponent is a corresponding growth in the number of terms in several of the governing equations. Two digital computer programs were developed to cope with this problem [6]. One, using a symbolic programming language called Reduce [8] (which is a sublanguage of the Lisp family), developed those governing equations which change with the creep exponent. The other solved the complete set of governing equations by using standard numerical analysis techniques. The second program was written in Fortran *H*.

The case in which the exponent is taken to be equal to five was also worked by hand [6], and as such, required the same simplifying assumptions that were used in the original "n-equal-to-3" solution. Both of these solutions were compared to the numerical solutions which did not involve the same assumptions.

Numerical solutions were obtained for the cases in which the exponent had the values 7 and 9 [6]. These four solutions formed the basis of two interpolation procedures by which the critical time may be obtained for any value of the creep exponent between 3 and 9. If a reasonable estimate is required, the critical time for cases in which the creep exponent lies between 9 and 29, may be obtained by extrapolating the results for the lower values of n.

THE ANALYTICAL RESULTS

Figure 1 indicates how the shell is loaded in compression. In order to simplify the analysis a sandwich model is substituted for the real shell wall, see Fig. 2. This model has been used successfully by several authors [9–11]. The details of the analysis are available for the cases in which the creep exponent is taken to be 3 [5] and 5 [6]. Only the results of the analyses will be presented here. Before this can be done however, certain notation should

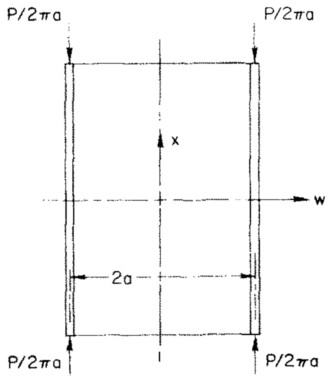

FIG. 1. Section through the thin cyclinder.

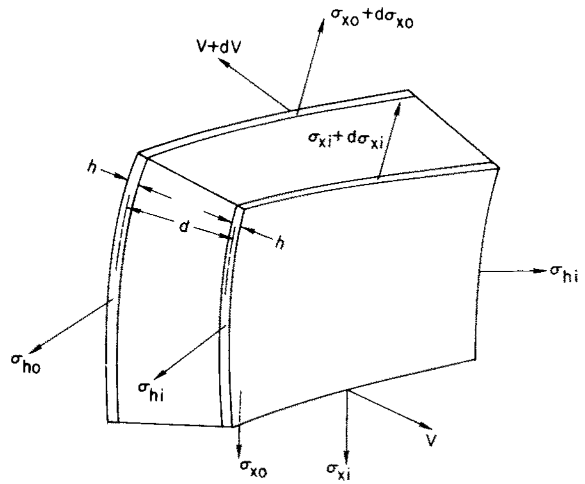

FIG. 2. The sandwich model.

be defined. The radial displacement w of the buckles is given by

$$w = f_0+f_1 \cos \alpha \tag{1}$$

where f_0 and f_1 are constants, and

$$\alpha = \pi x/\lambda \tag{2}$$

in which x is the axial measure and λ is the half-wavelength of the buckle. The axial stress, σ_{xi}, in the inner face of the sandwich model is given by

$$\sigma_{xi} = -C_0 - \sum_{j=1,2,\ldots}^{\infty} C_j \cos \alpha \tag{3}$$

in which C_0 and C_j are constants. The circumferential stress, σ_{hi} in the inner face of the sandwich model is given by

$$\sigma_{hi} = D_0 + \sum_{j=1,2,\ldots}^{\infty} D_j \cos \alpha. \tag{4}$$

The corresponding stresses σ_{x0} and σ_{h0} in the outer face are given by similar expressions, except that the constants are starred.

In order to understand the Euler time t_E, recall the classical elastic buckling stress for a thin cylinder

$$\sigma_{cr} \approx 0{\cdot}6E(t^*/a) \tag{5}$$

in which t^* is the wall thickness, and a the shell radius. Rabotnov† [12] has suggested a relationship between t^*, the wall thickness of the real shell, and h and d (see Fig. 2) characterizing the equivalent sandwich wall:

$$d = t^*[n/(2n+1)]^{n/(n+1)}, \qquad h = t^*/2 \tag{6}$$

† Rabotnov's suggestion insures that the creep rate is the same for the real shell and for its sandwich equivalent in the two limiting cases of pure compression and pure bending.

in which n is the creep exponent. Thus if n is taken as being equal to five, and if the classical stress is divided by the modulus, the result is the Euler strain ε_E:

$$\varepsilon_E = (\sigma_{cr}/E) \approx 1{\cdot}16(d/a). \tag{7}$$

And if the Euler strain is divided by the nominal creep strain rate $\dot{\varepsilon}_{nom}$, (which occurs at the onset of the creep process, and which is proportional to σ^n), the Euler time results:

$$t_E = \varepsilon_E/\dot{\varepsilon}_{nom}, \qquad \dot{\varepsilon}_{nom} = k\sigma^n. \tag{8}$$

A result of the original analysis [5], in which the creep exponent n is taken as equal to three is:

$$t/t_E = \tau = 0{\cdot}294 \ln\left[\frac{y_f^2(1{\cdot}18+y_i^2)}{y_i^2(1{\cdot}18+y_f^2)}\right] \tag{9}$$

in which ln indicates the natural logarithm and the subcripts f and i refer to final and initial values of the displacement parameter y which is dimensionless, and which is obtained by dividing f_1 by d. The critical time is defined as the time required for the deformations to exceed all bounds. By setting $y_f = \infty$, equation (9) becomes

$$t_{cr}/t_E = \tau_{cr} = 0{\cdot}294 \ln[(1{\cdot}18+y_i^2)/y_i^2]. \tag{10}$$

In most cases, since $y_i^2 \ll 1{\cdot}18$, this result may be simplified as follows:

$$\tau_{cr} = 0{\cdot}588 \ln(1{\cdot}08/y_i). \tag{11}$$

In the case in which n is taken to be equal to five [6], the results corresponding to equations (10) and (11) are:

$$\tau_{cr} = 0{\cdot}0555 \ln[(1{\cdot}25+y_i^4)/y_i^4] \tag{12}$$

and

$$\tau_{cr} = 0{\cdot}222 \ln(1{\cdot}06/y_i). \tag{13}$$

FURTHER SOLUTIONS

It was suggested [6] that equations (11) and (13) could be written in a general form:

$$\tau_{cr} = C_1 \ln(C_2/y_i) \tag{14}$$

where C_1 and C_2 are constants whose value depends on the creep exponent n. If it is assumed that the form of the expression for the critical time is independent of the value of the creep exponent, and that the values of C_1 and C_2 vary continuously with the creep exponent, then the problem of determining the critical time for a given creep exponent becomes the question of finding the particular value of the constants.

While the value of the creep exponent may be as high as 216 for the aluminum alloy 35-H12 at 90°F [13], most engineering materials exhibit creep exponents of less than 35 when tested at temperatures greater than 300°F. Thus it would appear that the values of the constants C_1 and C_2 need to be defined over a relatively small range of values of the creep exponent. However, a cursory comparison of the "n-equals-3" solution [5] and the "n-equals-5" solution [6] will indicate that a hand calculation for the cases in which n is greater than 5 would require an unreasonable amount of effort. In addition, Figs. 3 and 4 illustrate how the analytical assumptions degrade the value of the critical time solution as n increases from 3 to 5.

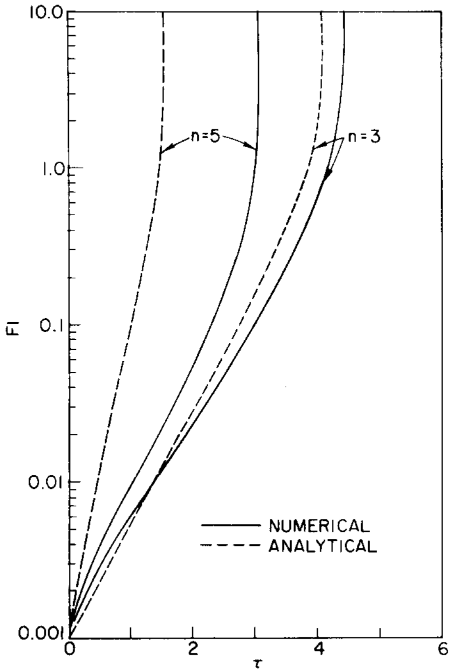

FIG. 3. Dimensionless displacement $F1$ as a function of dimensionless time τ.

These reasons, and the increased probability of bookkeeping error as the number of terms is increased prompted the development of two computer programs. The first, written in Reduce [15], (a sub-language in the Lisp [14] class), is able to produce the governing equations, in symbolic form, for any integral value of the creep exponent, and the second, using Fortran-*H*, solves the equations numerically.

THE FORTRAN PROGRAM

The ten governing equations are listed below. The first six are generated by the Reduce program, by combining the constitutive law, three conditions of compatibility and four stress expressions which are similar to equations (3) and (4). The resulting equations are true for all values of cos α, and this allows the equations to be split into sections which are multiplied by cos α, and those which are not.

$$\dot{\varepsilon}_{h0}-\dot{\varepsilon}_{hi} = 0 \quad \text{constant terms,} \tag{15}$$

$$\dot{\varepsilon}_{h0}-\dot{\varepsilon}_{hi} = 0 \quad \text{cosine terms,} \tag{16}$$

$$\dot{\varepsilon}_{x0}-\dot{\varepsilon}_{xi} = 0 \quad \text{constant terms,} \tag{17}$$

$$\dot{\varepsilon}_{x0}-\dot{\varepsilon}_{xi} = R\dot{\varepsilon}_{hi} \quad \text{cosine terms,} \tag{18}$$

$$(a/d)\dot{\varepsilon}_{hi} = \dot{f}_0 \quad \text{constant terms}, \tag{19}$$

$$(a/d)\dot{\varepsilon}_{hi} = \dot{f}_1 \quad \text{cosine terms}. \tag{20}$$

The subscripts h and x refer to hoop and axial directions respectively, while 0 and i indicate the outer and inner faces of the sandwich model. The remaining four equations are independent of n.

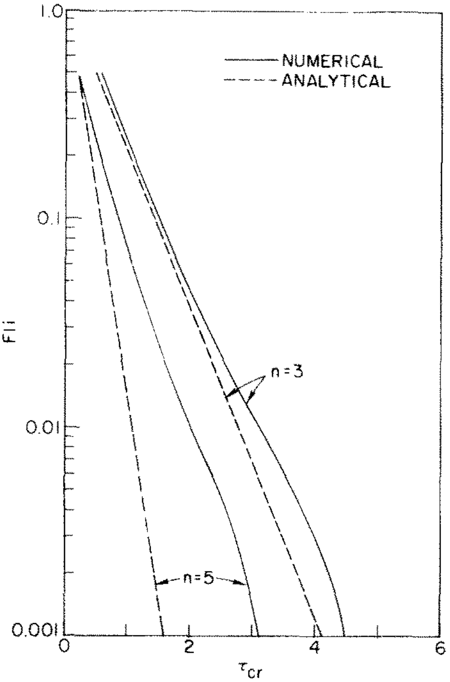

FIG. 4. Dimensionless critical time τ_{cr} as a function of initial value of dimensionless displacement $F1_i$.

$$C_0^* = 2\sigma - C_0, \tag{21}$$

$$C_1^* = -C_1, \tag{22}$$

$$D_0^* = -D_0, \tag{23}$$

and

$$D_1^* = -D_1 - RC_1 + 2pR(f_1/d), \tag{24}$$

where σ is the applied stress, and R is a dimensionless wavelength parameter. These equations are rendered dimensionless by dividing the stress coefficients by σ and the displacement coefficients by d.

The numerical solution is obtained in a stepwise manner. An initial value of f_1 is chosen, thus characterizing the initial imperfection in the shell. The stresses are then found by solving equations (15)–(18) simultaneously. This requires linearization of the equations, which is accomplished by use of the leading terms in four-dimensional Taylor series expansion [6]. The Taylor series, in turn, requires the partial derivatives of the equations, with respect to each unknown. These partials were developed by the Reduce program. The values of the four unstarred stress coefficients are found by Newton's method [16], and the starred coefficients are evaluated from equations (21)–(24). With the eight stress coefficients known, the remaining two equations (19) and (20) are solved. Equation (20) was integrated by using Simpson's Rule [16], and equation (19) was integrated by a graphical technique. The result of integrating equation (20) is the dimensionless time $\tau = t/t_E$, required for the displacement to grow from the initial value of f_1 to the new value chosen in Simpson's rule. This new value is then used to restart the cycle. The time is totaled after each cycle, and when the time increment reaches a sufficiently small value, the program is stopped. The total time to that point is taken to be the critical time.

RESULTS AND CONCLUSIONS

The computer programs were run for four values of the creep exponent, 3, 5, 7 and 9. Comparisons between the numerical and analytical results were made, and are presented in Figs. 3 and 4. In both cases it is clear that as n increases the analytical and numerical results diverge. The divergence is a consequence of the retention of only the first and last terms of the nth power of a binomial in the analytical solution; in the computer solution all the terms were considered. Evidently the accuracy of this analytical approximation decreases with increasing n. However, the integration could not be carried out in closed form without the simplification.

The effect of the creep exponent is clearly seen in Fig. 5, which indicates the behavior of the dimensionless displacement coefficient $F1$ for the four values of n; 3, 5, 7 and 9. These

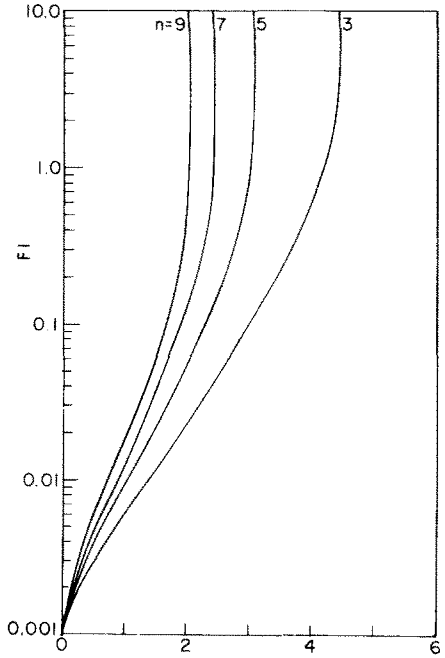

FIG. 5. Dimensionless displacement $F1$ as a function of dimensionless time τ, with creep exponent n as parameter ($F1_i = 0{\cdot}001$).

curves are for a given value of the initial value of $F1$, namely 0·001. Similar curves may be obtained for different initial values by either using the different values in the numerical procedure, or, approximately, by a simple graphical method, which is suggested by Fig. 6. In this figure, the value of n is constant, and equal to 3, however the initial value of $F1$ is varied. The shape of each curve is unchanged, and to obtain one curve from another requires only a horizontal shift of the original curve until it intersects the ordinate at the required initial value of $F1$. Of course the original curve must be produced from analytical or numerical data, and these data should relate to the smallest initial value of $F1$ that will be needed. This shifting property was checked by numerical experiment, and has been proved analytically [6] under certain restrictive conditions.

The effect of the wavelength parameter, R, was investigated numerically, and Fig. 7 shows how the critical time depends on R. It was noticed that the accuracy with which the most dangerous value of R can be analytically obtained, decreases as n is increased. In the case in which n equals 3, the most dangerous value of R was analytically predicted to be 2·0, while the numerical solution was 2·1. However in the case in which n equals 5, the analysis predicted 3·88 while the numerical result was again 2·1. It is assumed that the reason for this degradation is also due to the neglect of a larger number of terms when the small and large displacement assumptions are made.

Equation (14) was suggested as a means of writing a general solution for the critical time problem. The constants C_1 and C_2 were obtained analytically, and C_1 was corrected in accordance with the numerical solutions. The second constant C_2 was not adjusted since as it appears in the argument of a logarithm, its effect is reduced. By extrapolating the data

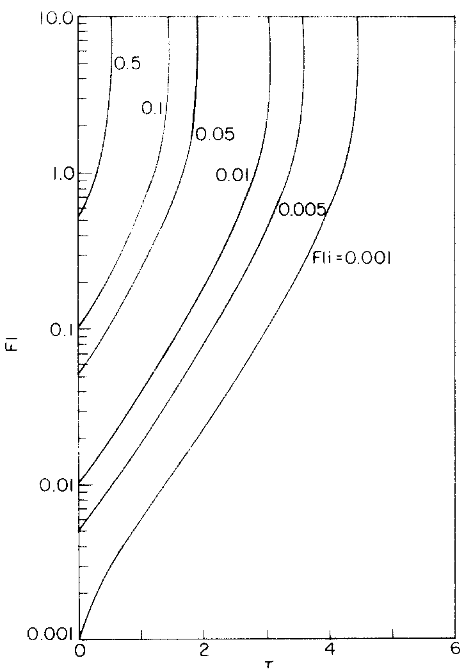

FIG. 6. Dimensionless displacement $F1$ as a function of dimensionless time τ, with initial value of dimensionless displacement $F1_i$ as parameter.

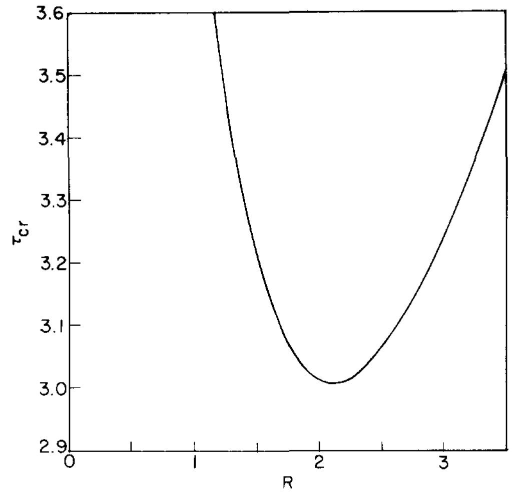

FIG. 7. Dimensionless critical time τ_{cr} as a function of dimensionless wavelength parameter R.

on C_2, new values of C_1 were obtained from the critical times for the cases in which n equals 7 and 9. Again, since C_2 is relatively ineffective, it may be extrapolated without undue concern for the accuracy. Each constant is now known for four values of n. If the curves are extended so that the length of the abscissa is approximately doubled, they will, because of the semi-logarithmic nature of the plot, yield data up to n equals 29. Such a plot is shown in Fig. 8.

The analytical accuracy of this technique is not defended, rather the results are offered as having perhaps some engineering usefulness, since the scatter in creep tests can be as high as 50 per cent, or more. The curves were used to develop critical times for a wide variety of cases, and the results are presented in Fig. 9. Since the parameters are dimensionless, the results may be applied to a wide spectrum of configurations.

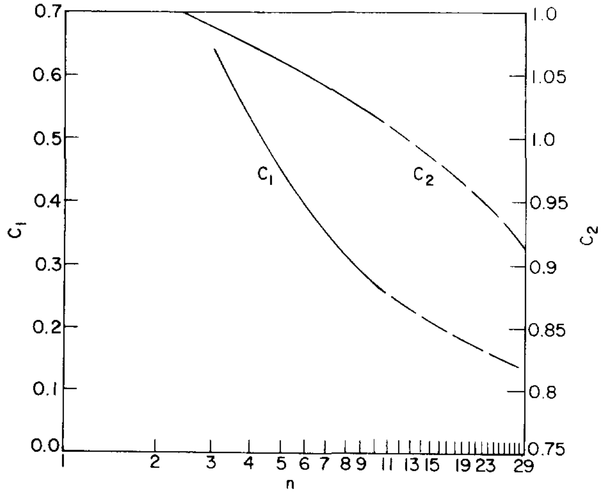

FIG. 8. Constants C_1 and C_2 as functions of creep exponent n.

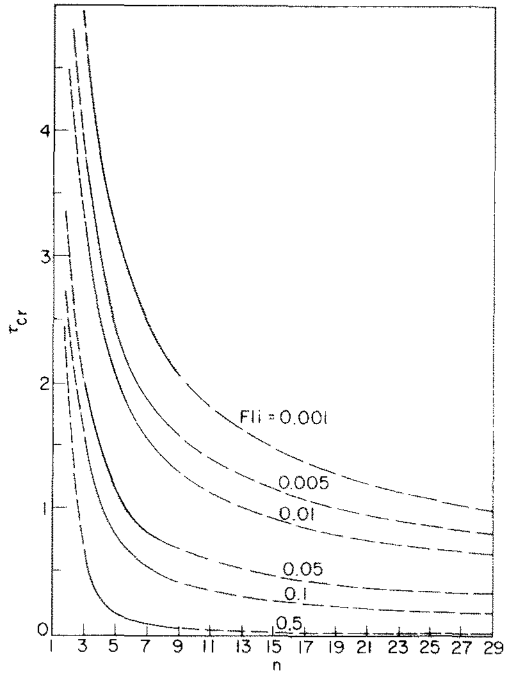

FIG. 9. Dimensionless critical time τ_{cr} as a function of creep exponent n, with initial value of dimensionless displacement $F1_i$ as parameter.

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Абстракт—Задача осесимметрической потери устойчивости ирц ползучести тонких, круглых, цилиндрических оболочек, подверженных действию равномерного осевого сжатия, решена ранее Хоффом. В этом решении определяющее уравнение степенного типа, с экспонентом равном три. Предметом предлагаемой работы является обобщение решения для более широкого круга значений экспонента ползучести *n*. Для обеспечения роста алгебраической комплексности, используется Ц В М в двух направлениях: для обобщения символически системы уравнений, и затем, для решения этих уравнеий. Используются численные программы для обобщения численных решений, для случаев, когда *n* равняется три, пять, семь и девять. Затем, приинммаются простые способы экстраполяции для получения решений задачи критического времени, для значений от *n* до 29.